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Frank H. Clarke and J.-P. Aubin

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MONOTONE INVARIANT SOLUTIONS TO DIFFERENTIAL INCLUSIONS

Frank H. Clarke[†] and J.-P. Aubin[‡]

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ABSTRACT

Let a given set be endowed with a preference preordering, and consider the problem of finding a solution to the differential inclusion

$$\dot{x}(t) \in S(x(t))$$

which remains in the given set and evolves monotonically with respect to the preordering. We give sufficient conditions for the existence of such a trajectory, couched in terms of a notion of tangency developed by Clarke. No smoothness or convexity is involved in the construction, which uses techniques of Filippov.

AMS (MOS) Subject Classifications: Primary 49E10; Secondary 34H05, 34A10

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Work Unit Number 1 (Applied Analysis)

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MONOTONE INVARIANT SOLUTIONS TO DIFFERENTIAL INCLUSIONS

Frank H. Clarke† and J.-P. Aubin‡

1. Statement of the problem

Let X be a compact subset of R^n, S a multifunction (set-valued function) from X into R^n with non-empty compact values. We regard X as the state set of a dynamical system and $S(x)$ as the set of feasible velocities of the system when its state is x. We introduce a preordering "$y \succ x$" (y is better than x) on X (i.e., a relation which is both reflexive and transitive).

Let $[0,T]$ be any finite interval $(T > 0)$. We say that an absolutely continuous function x from $[0,T]$ into R^n is a "monotone invariant trajectory for S starting at $x_0 \in X$" if

$$(1.1)\quad \begin{cases} \text{i)} & \dfrac{dx}{dt} \in S(x(t)) \quad \text{for almost all } t \text{ in } [0,T] \\ \text{ii)} & x(0) = x_0 \\ \text{iii)} & x(t) \in X \quad \text{for all } t \in [0,T] \\ \text{iv)} & \text{if } t \geq s,\ x(t) \text{ is better than } x(s)\,. \end{cases}$$

Our main result which we state in this section, gives reasonable sufficient conditions implying the existence of at least one monotone invariant trajectory. In §2 we discuss a notion of tangency for arbitrary

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closed sets [5] which is central in our results, while § 3 is devoted to the proof of the main theorem. We define the set

$$P(x) = \{y \in X \text{ such that } y \succ x\}$$

of elements y better than x. The multifunction P satisfies the properties:

$$(1.2) \qquad \begin{cases} \text{i) } \forall x \in X, \ x \in P(x) \\ \text{ii) } \forall x \in X, \forall y \in P(x), \ P(y) \subset P(x) . \end{cases}$$

(Conversely, if P is a multifunction from X to X satisfying (1.2), the relation $y \succ x$ defined by $y \in P(x)$ is a preordering on X.) We recall that P is said to be Lipschitz if and only if there exists $L > 0$ such that $P(x) \subset P(y) + L\|x - y\|B$, where B is the unit ball. We recall also that S is upper (resp. lower) semicontinuous if for any $\varepsilon > 0$, there exists a neighborhood $N_\varepsilon(x)$ of x such that,

$$\forall y \in N_\varepsilon(x), \ S(y) \subset S(x) + \varepsilon B \quad (\text{resp. } S(x) \subset S(y) + \varepsilon B) .$$

We will define in the next section the <u>tangent cone</u> $T(P(x);x)$ to the set $P(x)$ at x.

<u>Theorem 1</u>. <u>Let us assume that</u> P <u>satisfies</u> (1.2), <u>that</u> X <u>is compact</u>, <u>that the images</u> $S(x)$ <u>and</u> $P(x)$ <u>are non-empty and compact and that</u>

$$\begin{cases} \text{i) } S \text{ is continuous} \\ \text{ii) } P \text{ is Lipschitz.} \end{cases}$$

<u>Let us suppose also that</u>

$$(1.3) \qquad \forall x \in X, \ S(x) \subset T(P(x);x) .$$

Then, for any $x_0 \in X$, $v_0 \in S(x_0)$, and $T > 0$, there exists a monotone invariant trajectory starting at x_0 and satisfying

$$\begin{cases} \text{i)} & \dfrac{dx}{dt}(0) = v_0 \\ \text{ii)} & \dfrac{dx}{dt} \text{ is regulated (i.e., is a uniform limit of step functions)} . \end{cases}$$

Remark 1.1. In [2], the following was proved: if $X, S(x)$ and $P(x)$ are non-empty, convex and compact, if S is upper semicontinuous and P is continuous, and if $S(x) \cap T(P(x);x) \neq \phi$ for all $x \in X$, then there exists a monotone invariant trajectory.

Remark 1.2. If we neglect the monotonicity requirement (1.1) (iv), (i.e. set $P(x) = X$ for each x) a solution of (1.1) (i), (ii), (iii) is called an invariant trajectory. When S is a continuous function, existence of invariant trajectories was obtained by Nagumo [12], Crandall [7], Martin [10], [11], Yorke [15], Hartman [9]; for S a Lipschitz function, see Brezis [4], Bony [3], Redheffer [14]; for S a Lipschitz multifunction see Clarke [5]. Theorem 1 implies the existence of (at least) one invariant trajectory when S is a continuous multifunction; local existence of the differential inclusion (1.1) (i), (ii) was proved by Filippov [8]. We make use of the techniques introduced in this paper. See also Antosiewicz-Cellina [1], Olech [13] and the references of the latter paper.

2. Tangent Cones

We recall that any locally Lipschitz function $f : R^{\ell} \to R$ has a generalized directional derivative, defined in [5], [6] by

$$(2.1) \qquad f^0(x;v) = \limsup_{\substack{y \to x \\ \theta \to 0+}} [f(y + \theta v) - f(y)]/\theta$$

which is convex positively homogeneous and continuous with respect to v. It coincides with $\langle \nabla f(x), v \rangle$ if f is continuously differentiable and with the derivative from the right if f is convex and continuous. By definition, the generalized gradient $\partial f(x)$ of f at x is the set of $p \in R^{\ell *}$ such that

$$\langle p, v \rangle \leq f^0(x;v) \quad \text{for all } v \text{ in } R^{\ell}.$$

It is a non-empty convex compact set whose support function is $f^0(x;\cdot)$. Let X be a closed subset of R^{ℓ}. The distance function $y \mapsto d(X;y) = \inf_{z \in X} \|y - z\|$, being a Lipschitz function, admits a generalized directional derivative $d^0(X;y;v)$ and a generalized gradient $\partial d(X;y)$ at any point y. If $y \in X$, we shall say that the normal cone $N(X;y)$ to X at y is the closed convex cone spanned by $\partial d(X;y)$. We define the tangent cone $T(X;y)$ to X at y to be the (negative) polar cone of $N(X;y)$. Thus

$$(2.2) \qquad T(X;y) = \{v \in R^{\ell} \text{ such that } d^0(X;y,v) \leq 0\}.$$

If the interior $\operatorname{int} X$ of X is non-empty and if $y \in \operatorname{int} X$, then $T(X;y) = R^{\ell}$. When X is a closed convex set, or a smooth manifold, these definitions coincide with the usual ones (see [5]). We recall the following characterization of the tangent cone (see [5, p. 256]).

Proposition 1. A vector v belongs to $T(X;y)$ if and only if

$$\lim_{\substack{z \in X \\ z \to y}} \liminf_{\lambda \to 0+} d(X;z + \lambda v)/\lambda = 0 . \tag{2.3}$$

This implies obviously that

$$\liminf_{\lambda \to 0+} d(X;y + \lambda v)/\lambda = 0 , \tag{2.4}$$

but this condition does not necessarily imply that $v \in T(X;y)$.

Remark 2.1. It is easy to show from the above (see [5, Proposition 3.7]) that the tangent cone may be defined directly as follows: v belongs to $T(X;x)$ iff for every sequence $\{x_n\}$ in X converging to x and every sequence $\{\lambda_n\}$ of positive numbers converging to 0, there exists a sequence $\{v_n\}$ converging to v such that $x_n + \lambda_n v_n$ belongs to X infinitely often.

Lemma 1. For any $x_0 \in X$ and $v_0 \in R^{\ell}$, there exist neighborhoods $N(x_0)$ of x_0, $M(v_0)$ of v_0 and $h(x_0, v_0) > 0$ such that

$$d(X;x + hv)/h \leq d^0(X;x_0;v_0) + \varepsilon \tag{2.5}$$

when $x \in N(x_0)$, $v \in M(v_0)$ and $0 < h \leq h(x_0, v_0)$.

Proof. Let us set $g(t) = d(X;x + tv)$. Since $g(0) = d(X;x) = 0$ when $x \in X$, we can write

$$d(X;x + hv)/h = \frac{1}{h} \int_0^h g'(t)dt . \tag{2.6}$$

[Indeed g, being a Lipschitz function of t, is differentiable almost everywhere]. But we have

$$(2.7) \qquad g'(t) \le d^0(X;x+tv;v)$$

since, if g is differentiable at t

$$g'(t) = \lim_{\theta \to 0+} [d(X;x+tv+\theta v) - d(X;x+tv)]/\theta$$

$$\le \limsup_{\substack{y \to x+tv \\ \theta \to 0+}} [d(X;y+\theta v) - d(X;y)]/\theta = d^0(X;x+tv;v) .$$

Since the function $(y,v) \mapsto d^0(X;y;v)$ is obviously upper semicontinuous, we can associate with any $\varepsilon > 0$, a neighborhood $N(x_0)$ of x_0, $M(v_0)$ of v_0 and a number $h(x_0,v_0) > 0$ such that when $x \in N(x_0)$, $v \in M(v_0)$ and $t \le h(x_0,v_0)$, then

$$(2.8) \qquad d^0(X;x+tv;v) \le d^0(X;x_0;v_0) + \varepsilon .$$

The result then follows from this, combined with (2.7) and (2.6). Q.E.D.

Let a multifunction S from X to R^ℓ be given. We shall now consider various consequences of the hypothesis that $S(x)$ is contained in $T(X;x)$ for each x (this is (1.3) in the case $P(x) = X$ for each x; the connection to Theorem 1 is made in Proposition 4 below). Besides developing some machinery that will be needed in § 3, the relationships between the various hypotheses made in the papers on invariant trajectories cited above will be clarified (see Corollary 1).

<u>Proposition 2</u>. <u>Let us assume that</u> S <u>is upper semicontinuous and that</u>

$$\forall x \in X, \; S(x) \subset T(X;x) .$$

<u>Then the function</u>

$$a(x,h) = \sup_{v \in S(x)} d(X;x+hv)/h$$

<u>converges to</u> 0 <u>with</u> h <u>uniformly for</u> x <u>in compact subsets</u> $(h > 0)$.

<u>Proof</u>. Let K be a compact subset of X. Since S is upper semicontinuous, the graph $G_K(S)$ of the restriction of S to K is compact. Consequently, it can be covered by a finite number p of neighborhoods $N(x_i) \times M(v_i)$, where $(x_i, v_i) \in G_K(S)$ and the neighborhoods N and M have the properties of the preceding lemma. If we set $h(K,\varepsilon) = \min_{i=1,\ldots,p} h(x_i, v_i)$, we deduce that

$$\forall x \in K,\ \forall v \in S(x),\ \forall h < h(K,\varepsilon),\ d(X;x+hv)/h \leq \varepsilon .$$

So the proposition is proved. Q.E.D.

For any $y \in R^{\ell}$, let $\pi(y)$ denote the set of closest points in X to y.

<u>Proposition 3</u>. <u>If we assume that</u>

$$\forall x \in X,\ \forall v \in S(x),\ \liminf_{h \to 0+} d(X;x+hv)/h = 0 , \tag{2.9}$$

<u>then</u>

$$\forall y \in R^{\ell},\ \forall x \in \pi(y),\ \forall v \in S(x),\ \langle y-x, v\rangle \leq 0 . \tag{2.10}$$

<u>Conversely, if we assume that</u> S <u>is lower semicontinuous property</u> (2.10) <u>implies that</u> $S(x) \subset T(X;x)$ <u>for all</u> $x \in X$.

<u>Remark</u>. There are examples where (2.10) does not imply (2.9).

<u>Proof</u>. Suppose that (2.10) is false. There exist $y \in R^{\ell}$, $x \in \pi(y)$ and $v \in S(x) \cap T(X;x)$ such that $\langle y-x, v\rangle > 0$. Let C be the complement of the open ball of center y and radius $d(X;y)$. Then

$d(X;x + tv) \geq d(C;x + tv)$ and, since $\langle y - x, v\rangle > 0$, $d(C;x + tv) \geq \delta t + o(t)$ where $\delta > 0$. Hence $\liminf_{\theta \to 0+} d(X;x + \theta v)/\theta \geq \delta > 0$; this contradicts the fact that $v \in T(X;x)$. Conversely, let us prove that if S is continuous, then (2.10) implies that $S(x) \subset T(X;x)$. By [5, p. 254], we know that any element of $N(X;x)$ can be written as a convex combination of elements $p = \lim_{n \to \infty} s_n(y_n - z_n)$ where $s_n > 0$, $y_n \to x$ and $z_n \in \pi(y_n)$. Since $x \in X$, we have $\|x - z_n\| \leq \|x - y_n\| + \|y_n - z_n\| \leq 2\|x - y_n\|$. Since S is lower semicontinuous, any $v \in S(x)$ is the limit of some sequence of elements $v_n \in S(z_n)$. So $\langle p, v\rangle = \lim_{n \to \infty} s_n\langle y_n - z_n, v_n\rangle \leq 0$. This implies that $v \in N(X;x)^- = T(X;x)$. ■

<u>Corollary 1</u>. <u>Let us assume that</u>

$$
(2.11) \qquad \begin{cases} \text{i) } X \text{ is a compact set.} \\ \text{ii) } S \text{ is a continuous multifunction with non-empty} \\ \quad\; \text{compact values.} \end{cases}
$$

<u>Then the following statements are equivalent</u>

(a) $\forall x$, $S(x) \subset T(X;x)$

(b) $\forall x \in X$, $\forall v \in S(x)$, $\lim_{\substack{y \to x \\ y \in X}} \liminf_{\theta \to 0+} d(X;y + \theta v)/\theta = 0$

(c) $\lim_{h \to 0} \sup_{\substack{x \in X \\ v \in S(x)}} d(X;x + hv)/h = 0$

(d) $\forall x \in X$, $\forall v \in S(x)$, $\lim_{h \to 0+} d(X;x + hv)/h = 0$

(e) $\forall x \in X$, $\forall v \in S(x)$, $\liminf_{h \to 0+} d(X;x + hv)/h = 0$

(f) $\forall y \in R^{\ell}$, $\forall x \in \pi(y)$, $\forall v \in S(x)$, $\langle y - x, v\rangle \leq 0$.

Proof. That each condition implies the next, and that (f) implies (a), is either self-evident or a consequence of the preceding results. Q. E. D.

Remark 2.2. Conditions (a) and (b) are found in [5], while (d) appears in [4], [7], [9], (e) in [14], [15], [12] and (f) in [3], [7].

Let us consider now another multifunction P mapping X into X.

Proposition 4. Let X be a compact subset of R^n.

Let us assume that S is a continuous multifunction with non-empty compact values and that P is a closed multifunction satisfying (1.2). If we assume

$$\forall x \in X, \; S(x) \subset T(P(x);x) ,$$

then

$$\lim_{h \to 0+} \sup_{\substack{x \in X \\ v \in S(x)}} d(G(P);(x, x + hv))/h = 0$$

where $d(G(P);(\cdot,\cdot))$ is the distance to the graph $G(P)$ of P.

Proof. Let $\tilde{S} : G(P) \to R^{2\ell}$ be the continuous multifunction defined as follows:

$$\tilde{S}(x, w) = \{0\} \times S(w) .$$

Let (x, w) be a point in $G(P)$, and let $(0, v)$ belong to $\tilde{S}(x, w)$. We claim that

$$\liminf_{h \to 0+} d(G(P);(x, w + hv))/h = 0 . \tag{2.12}$$

To see this, note that since x belongs to $P(x)$,

$$d(G(P);(x, w + hv)) \leq d(P(x);w + hv) .$$

Also, $w \in P(x)$ implies $P(w) \subset P(x)$, so that in turn

$$d(P(x);w + hv) \leq d(P(w);w + hv) .$$

However,

$$\liminf_{h \to 0+} d(P(w);w + hv)/h = 0$$

by (2.4), since $v \in S(w) \subset T(P(w);w)$. Thus (2.12) ensues. We now apply Corollary 1 (with X replaced by $G(P)$ and S by $\tilde{S}$) and deduce that condition (c) of that result holds. But that implies the desired result, since $(x, x) \in G(P)$. Q.E.D.

3. Proof of the Theorem

By Proposition 4, we know that our assumptions imply that

$$b(h) = \sup_{\substack{x \in X \\ v \in S(x)}} d(G(p);(x, x + hv))/h$$

converges to 0 with h. We will use this fact instead of assumption (1.3) in the proof of the theorem.

We consider a decreasing sequence of partitions $\mathcal{P}(h_m)$ of $[0, T]$ made of intervals $[qh_m, (q+1)h_m]$ where q is an integer and where

$$(3.1) \qquad T/h_1 \text{ and } \frac{h_{m-1}}{h_m} \text{ are integers .}$$

We shall denote by $\tau_m = qh_m$ (where $q \in N$) any node of the partition $\mathcal{P}(h_m)$. We shall construct a sequence of piecewise linear functions x_m on the partition $\mathcal{P}(h_m)$ defined by

$$(3.2) \qquad \forall t \in [\tau_m, \tau_m + h_m],\ x_m(t) = x_m(\tau_m) + (t - \tau_m)v_m(\tau_m)$$

whose derivative $\dot{x}_m(t)$ is a step function:

$$(3.3) \qquad \forall t \in [\tau_m, \tau_m + h_m[,\ \dot{x}_m(t) = v_m(\tau_m).$$

For that purpose, we define the map $j : \tau_m \to j(\tau_m)$ associating with any node $\tau_m \in \mathcal{P}(h_m)$ the smallest index $j = j(\tau_m)$ such that $\frac{\tau_m}{h_{j+1}}$ is an integer. [Indeed, if $1 \leq k \leq m$ is such that $\frac{\tau_m}{h_k}$ is an integer, then $\frac{\tau_m}{h_\ell}$ is also an integer if $k \leq \ell \leq m$]. We also define the map $\psi : \tau_m \to \psi(\tau_m)$ associating with any positive node $\tau_m \in \mathcal{P}(h_m)$ the largest node $\psi(\tau_m)$ of the partition $\mathcal{P}(h_{j(\tau_m)})$ strictly smaller than τ_m (if $j = 0$, $\psi(\tau_m) = 0$). We know that $S(X)$ is compact (and thus, contained in a ball of radius $c - 1 > 0$). Since S is uniformly continuous, we can choose δ_m such that

(3.4) $$\|x - y\| \leq c\delta_m \text{ implies that } S(y) \subset S(x) + \varepsilon_m B ,$$

where we set $\varepsilon_m = 2^{-m}$. We denote by L the Lipschitz constant of P:

$$P(y) \subset P(x) + L\|x - y\|B .$$

We shall choose h_m small enough in order that

$$h_m \leq \delta_m, \; 2(L + 1)b(h_m) \leq \varepsilon_m ,$$

where

$$b(h) = \sup_{(x, v) \in G(S)} d(G(P);(x, x + hv))/h .$$

<u>Construction of x_m on $[0, h_m]$</u>. We consider $v_0 \in S(x_0)$. By definition of $b(h_m)$, there exist $(y, z) \in G(P)$ such that $\max(\|x_0 - y\|, \|x_0 + h_m v_0 - z\|) \leq 2b(h_m)h_m$. Since P is Lipschitz, then $z \in P(y) \subset P(x_0) + L\|x_0 - y\|B \subset P(x_0) + 2Lb(h_m)h_m B$. Therefore, there exists $u \in P(x_0)$ such that $\|z - u\| \leq 2Lb(h_m)h_m$ and thus, such that

$$\left\|\frac{x_0 - u}{h_m} + v_0\right\| \leq 2(L + 1)b(h_m) \leq \varepsilon_m .$$

If we define x_m on $[0, \tau_m]$ by setting

$$x_m(0) = x_0 \text{ and } v_m(0) = \frac{u - x_0}{h_m}$$

we obtain the properties

$$\begin{cases} \text{i)} & v_m(0) \in S(x_0) + \varepsilon_m \\ \text{ii)} & \|v_m(0) - v_0\| \leq \varepsilon_m \\ \text{iii)} & x_m(h_m) = u \in P(x_0) . \end{cases}$$

<u>Construction of x_m on $[\tau_m, \tau_m + h_m]$.</u> Let us assume that we have constructed x_m on $[0, \tau_m]$ satisfying, for any node $\sigma_m < \tau_m$,

$$(3.5) \qquad \begin{cases} \text{i)} & v_m(\sigma_m) \in S(x_m(\sigma_m)) + \varepsilon_m \\ \text{ii)} & \|v_m(\sigma_m) - v_m(\psi(\sigma_m))\| \leq 2\varepsilon_{j(\sigma_m)} \\ \text{iii)} & x_m(\sigma_m + h_m) \in P(x_m(\sigma_m)) . \end{cases}$$

We shall construct x_m on the interval $[\tau_m, \tau_m + h_m]$ satisfying properties (3.5) where σ_m is replaced by τ_m. Let us set $j = j(\tau_m) < m$ and $\tau_j = \psi(\tau_m) < \tau_m$. Since the node $\tau_j = \sigma_m$ is also a node σ_m of the partition $\mathcal{P}(h_m)$, we know that, by (3.5) (i),

$$(3.6) \qquad v_m(\tau_j) \in S(x_m(\tau_j)) + \varepsilon_m .$$

Furthermore, $\|x_m(\tau_m) - x_m(\tau_j)\| \leq \int_{\tau_j}^{\tau_m} \|v_m(t)\| dt \leq c|\tau_m - \tau_j| \leq ch_j$.

We deduce from the uniform continuity of S that (see (3.4))

$$(3.7) \qquad S(x_m(\tau_j)) \subset S(x_m(\tau_m)) + \varepsilon_j B .$$

Hence there exists $w \in S(x_m(\tau_m))$ such that

$$(3.8) \qquad \|v_m(\tau_j) - w\| \leq \varepsilon_j + \varepsilon_m .$$

Now, by definition of $b(h_m)$, there exists $(y, z) \in G(P)$ such that

(3.9) $$\max(\|x_m(\tau_m) - y\|, \|x_m(\tau_m) + h_m w - z\|) \leq 2b(h_m)h_m .$$

Therefore, since P is Lipschitz, we deduce that

(3.10) $$z \in P(y) \subset P(x_m(\tau_m)) + L\|x_m(\tau_m) - y\|B .$$

Then, there exists $u \in P(x_m(\tau_m))$ such that

$$\|z - u\| \leq L\|x_m(\tau_m) - y\| \leq 2Lb(h_m)h_m$$

and thus, such that

$$\left\|\frac{x_m(\tau_m) - u}{h_m} + w\right\| \leq 2(L + 1)b(h_m) \leq \varepsilon_m .$$

If we set

(3.10) $$v_m(\tau_m) = \frac{u - x_m(\tau_m)}{h_m}$$

we thus have shown that $v_m(\tau_m) \in w + \varepsilon_m B \subset S(x_m(\tau_m)) + \varepsilon_m B$, that $\|v_m(\tau_j) - v_m(\tau_m)\| \leq \|v_m(\tau_j) - w\| + \|w - v_m(\tau_m)\| \leq \varepsilon_j + 2\varepsilon_m \leq 2\varepsilon_j$ (since $j < m$) and that $u = x_m(\tau_m + h_m) \in P(x_m(\tau_m))$. So x_m is constructed on $[x_m, x_m + \tau_m]$ and satisfies properties (3.5) with $\sigma_m = \tau_m$.

<u>Convergence of the sequence of approximate solutions</u>. We shall prove now that the sequence $\{\dot{x}_m\}$ is totally bounded in the space $\mathcal{B}(0, T; R^\ell)$ of bounded functions from $[0, T]$ into R^ℓ. Let ε_k be fixed. Since $S(X)$ is compact, it can be covered by p balls $u_j + \varepsilon_k B$ where $u_j \in S(X)$. Let us consider any interval $[\tau_k, \tau_k + h_k[$ of the partition

$\mathcal{P}(h_k)$. There exists u_j^k such that

(3.11) $$\|u_j^k - \dot{x}_m(\tau_k)\| \leq \varepsilon_k \, .$$

If $m \leq k$, $\dot{x}_m(t) = \dot{x}_m(\tau_k)$ is constant on this interval. Let $m > k$ and $t \in [\tau_k, \tau_k + h_k[$. Then we shall prove that

$$\|\dot{x}_m(t) - \dot{x}_m(\tau_k)\| \leq 4\varepsilon_k \, .$$

Indeed, there exists a node $\tau_m \in \mathcal{P}(h_m)$ such that $t \in [\tau_m, \tau_m + h_m[\subset [\tau_k, \tau_k + h_k[$. If $\tau_m = \tau_k$, then $\dot{x}_m(t) = \dot{x}_m(\tau_k)$. If $\tau_k < \tau_m$, then there exists $j_1 = j(\tau_m)$ such that $k \leq j_1 < m$ and $\tau_k \leq \psi(\tau_m) < \tau_m$. If $k < j(\tau_m)$, then there exists $j_2 = j^2(\tau_m)$ such that $k \leq j_2 < j_1$ and $\tau_k \leq \psi^2(\tau_m) < \psi(\tau_m) < \tau_m$. Proceeding in this way, we can eventually write $\tau_k = \psi^{\ell}(\tau_m)$ where $\ell \leq m - k$. Hence, since $\dot{x}_m(t) = v_m(\tau_m)$ and $\dot{x}_m(\tau_k) = v_m(\psi^{\ell}(\tau_m))$, inequalities (3.11) imply that

(3.12) $$\|\dot{x}_m(t) - \dot{x}_m(\tau_k)\| \leq 2(\varepsilon_k + \varepsilon_{k+1} + \cdots + \varepsilon_m) \leq 4\varepsilon_k \, .$$

So, we deduce from (3.11) and (3.12) that for any m, for any $t \in [\tau_k, \tau_k + h_k]$, we have

(3.13) $$\|\dot{x}_m(t) - u_j^k\| \leq 5\varepsilon_k \, .$$

This implies that for any ε_k, each function $\dot{x}_m(t)$ is in a ball of radius $5\varepsilon_k$ whose center is a step function u_m such that $u_m(t) = u_j^k$ if $t \in [\tau_k, \tau_k + h_k[$. Since there is a finite number $(p \frac{T}{h_k})$ of such step functions we have proved that the sequence of derivatives $\dot{x}_m$ is totally bounded.

Consequently, we can extract a subsequence (still denoted by) $\dot{x}_m$ converging uniformly to a function $v \in \mathcal{B}(0,T;R^{\ell})$. Since $x_m(t) = x_0 + \int_0^t \dot{x}_m(\tau)d\tau$, the sequence x_m converges uniformly to a continuous function x such that

$$x(t) = x_0 + \int_0^t v(t)dt .$$

Therefore, for any $t \in [0,T]$, $x(t)$ and $v(t)$ are respectively the limits of sequences $x_m(\tau_m)$ and $\dot{x}_m(\tau_m) = v_m(\tau_m)$. So, by the upper semicontinuity of S, we deduce that

$$v(t) \in \dot{x}_m(\tau_m) + \frac{\varepsilon}{2} B \subset S(x_m(\tau_m)) + \frac{\varepsilon}{2} B \subset S(x(t)) + \varepsilon B$$

when m is large enough. Hence $v(t) = \dot{x}(t) \in S(x(t))$. Furthermore, $x(t) \in X$. Finally, property (1.2) (ii) of P, combined with (3.5) (iii), implies that in any of our partitions, larger nodes are better than smaller ones. We deduce from this that $P(x(t)) \subset P(x(s))$ when $t > s$. Thus we have proved the existence of a monotone trajectory. Q.E.D.

Remark 3.1. The above proof remains valid when X is a compact subset of a Banach space U, S is a continuous multifunction from X into U with non-empty compact images, P a Lipschitz multifunction from X into X satisfying (1.2) with non-empty compact images and $b(h)$ converges to 0 with h; under these assumptions, there exists a regulated monotone invariant trajectory satisfying $x(0) = x_0$ and $\dot{x}(0) = v_0 \in S(x_0)$.

Remark 3.2. We can generalize Theorem 1 to the case of time-dependent systems, by assuming that the multifunctions $S:[0,T]\times X \to U$ and $P:[0,T]\times X \mapsto X$ are continuous, that for any t, $x \mapsto P(t,x)$ is Lipschitz and satisfies (1.2), and that $S(t,x) \subset T(P(t,x);x)$ for any $(t,x) \in [0,T]\times X$. The above proof then needs no modifications. By using techniques of Olech [13], the case where $t \mapsto S(t,x)$ is measurable for any $x \in X$ and where $x \mapsto S(t,x)$ is continuous for almost all t can also be treated.

Remark 3.3. We can consider also the case where X is no longer compact, but closed. Let the intersection of X with the ball of center x_0 and radius d be denoted X_d, and associate with d the scalar

$$c(d) = \sup_{\substack{x \in X_d \\ v \in S(X)}} \|v\| + 1 ,$$

it is easy to check that the approximate solutions x_m satisfying (3.5) remain in X_d whenever $t \leq T(d) = d/c(d)$. Therefore, by replacing X by X_d in the proof of Theorem 1, we obtain the existence of a monotone trajectory which remains in X_d when $t \leq T(d)$.

Remark 3.4. If the images $S(x)$ are convex, we can replace assumption (1.3) by

$$(3.14) \qquad a(h) = \sup_{x \in X} \inf_{v \in S(x)} d(G(P);(x, x+hv))/h \to 0 \quad \text{as} \quad h \downarrow 0 .$$

We can prove that (3.14) holds whenever we assume

$$(3.15) \qquad \forall x \in X,\ S(x) \cap T(P(x);x) \neq \phi \quad \text{(cf. remark 1.1)} .$$

REFERENCES

1. H. A. Antosiewicz and A. Cellina, Continuous selections and differential relations, J. Diff. Eq. 19 (1975), 386-398.
2. J.-P. Aubin, A. Cellina, and J. Nohel, Monotone trajectories of multivalued dynamical systems, MRC Technical Report, University of Wisconsin-Madison, 1976.
3. J.-M. Bony, Principe du maximum, inégalité de Harnack et unicité du probléme de Cauchy pour les opérateurs elliptiques dégénerés, Ann. Inst. Fourier 19 (1969), 277-304.
4. H. Brézis, On a characterization of flow invariant sets, Comm. Pure Appl. Math. 23 (1970), 261-263.
5. F. H. Clarke, Generalized gradients and applications, Trans. Amer. Math. Soc. 205 (1975), 247-262.
6. F. H. Clarke, Generalized gradients of Lipschitz functionals, MRC Technical Report, University of Wisconsin-Madison, 1976.
7. M. G. Crandall, A generalization of Peano's existence theorem and flow invariance, Proc. Amer. Math. Soc. 36 (1972), 151-155.
8. A. F. Filippov, On the existence of solutions of multivalued differential equations (In Russian), Math. Zametki. 10 (1971), 307-313.
9. P. Hartman, On invariant sets and on a theorem of Wazewski, Proc. Amer. Math. Soc. 32 (1972), 511-520.
10. R. H. Martin, Differential equations on closed subsets of a Banach space, Trans. Amer. Math. Soc. 179 (1973), 399-414.

11. R. H. Martin, Approximation and existence of solutions to ordinary differential equations in Banach spaces, Funkcialaj. Ekuacioj. 16 (1973), 195-211.

12. M. Nagumo, Über die Laga der Integralkurven gewöhnlicher Differentialgleichungen, Proc. Phys. Math. Soc. Japan 24 (1942), 551-559.

13. C. Olech, Existence of solutions of nonconvex orientor fields, Boll. Unione Mat. Ital. 12 (1975), 189-197.

14. R. M. Redheffer, The theorems of Bony and Brezis on flow invariant sets, Amer. Math. Monthly 79 (1972), 790-797.

15. J. A. Yorke, Invariance for ordinary differential equations, Math. Syst. Theory 1 (1967), 353-372.

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Let a given set be endowed with a preference preordering, and consider the problem of finding a solution to ~~the~~ a differential inclusion

$$\dot{x}(t) \in S(x(t))$$

which remains in the given set and evolves monotonically with respect to the preordering. ~~We give~~ sufficient conditions are given for the existence of such a trajectory, couched in terms of a notion of tangency developed by Clarke. No smoothness or convexity is involved in the construction, which uses techniques of Filippov.

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